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The hydrodynamic interaction of two spheres moving in an unbounded fluid at small but finite Reynolds number

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The forces on two spherical particles moving in a fluid are investigated by the method of matched asymptotic expansions in the small Reynolds number, for the case when the particles are within each other's inner region of expansion. The particular case in which the distance l between the sphere centres is very much larger than the sphere radii a and b is studied in detail. The asymptotic expansion of the force on one of the spheres for small a/l and b/l is obtained. Some properties of the force, not to be expected from the Stokes equation, are revealed.

1. Introduction

Hydrodynamic interactions between particles have significant effects on the bulk properties of a fluid–particle system such as a suspension or a flow through a porous medium. They have been the subject of many studies for many years (for references see Happel & Brenner 1973; Batchelor & Green 1972). If particles are randomly located within a fluid, the most important hydrodynamic interactions are those between a pair of particles. So far most studies of the pair interaction have been based on the Stokes equation. However, they cannot fully explain interesting properties of the interaction. For example, in the case when two spheres of equal size are sedimenting vertically one above the other in an unbounded fluid, the difference between the forces on the leading and trailing spheres cannot be explained by an analysis based on the Stokes equation.

In order to investigate such properties, we must take into account the inertial effect. Let us consider two particles moving in an unbounded fluid at small but finite Reynolds number R. To treat the nonlinear inertial term in the Navier–Stokes equation properly, we have to use the method of matched asymptotic expansions. Then there can be at least the following two cases: case I, where the particles are sufficiently separated so that each of them is located in the outer region of expansion of the other; and case II, where they are sufficiently close to each other so that each of them is located in the inner region of expansion of the other. Case I was studied by Vassure & Cox (1977).

It is intuitively evident that the particles experience stronger interaction with a smaller separation between them. Hence the study of case II is expected to be crucial for understanding the effect of the particle–particle interaction on the

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hydrodynamical bulk properties of a particulate system with non-zero particle Reynolds number. The purpose of this paper is to study this case, i.e. case II.

Brenner & Cox (1963) presented a method of calculating the first order (in R) force on a particle of arbitrary shape in a uniform flow. Their argument can be applied to obtain the *total* force on two particles (in case II) but is insufficient to obtain the *individual* force on each of them. Their method is generalized in §§3–5 so that the individual forces can be calculated; for simplicity the particles are assumed to be spheres. (The discussion of §§3–5 can be easily extended to the case where the particles are of arbitrary shapes.) In §§6 and 7, confining ourselves to the case where the distance l between the spheres is very much larger than their radii a and b, we analyse the force on one of them. The asymptotic expansion of the force is given up to $O(a/l)$ or $O(b/l)$.

For full understanding of the R-dependence of the bulk property of a fluid–particle system, it would be necessary to study more general cases including the case with a/l and/or b/l being order unity. The approach in §§3–5 is applicable to such cases also. Such analyses are left to future studies.

2. Basic equations

We consider two spherical particles of radii a and b (which will be referred to respectively as sphere A and sphere B) moving in a quiescent unbounded incompressible fluid. The instantaneous translational and angular velocities of sphere A (sphere B) are denoted by $\mathbf{U}'_A$ and $\mathbf{\Omega}'_A$ ($\mathbf{U}'_B$ and $\mathbf{\Omega}'_B$) respectively. We choose a Cartesian coordinate system (r_1, r_2, r_3), the origin of which is at the instantaneous position of the centre of sphere A, and relative to which the fluid velocity at infinity is $-\mathbf{U}'_A$. In this coordinate system the flow may be regarded as a steady one, provided that the motions of the spheres are steady and $|\mathbf{U}'_A - \mathbf{U}'_B|$ is sufficiently small.

The fluid velocity $\mathbf{u}'$ and pressure p' (taken to be zero at infinity) then satisfy the steady Navier–Stokes and continuity equations

$$\mu \nabla'^2 \mathbf{u}' - \nabla' p' = \rho \mathbf{u}' . \nabla' \mathbf{u}', \tag{2.1a}$$

$$\nabla' . \mathbf{u}' = 0, \tag{2.1b}$$

and the boundary condition at infinity

$$\mathbf{u}' \rightarrow -\mathbf{U}'_A \quad \text{as} \quad |\mathbf{r}'| \rightarrow \infty, \tag{2.2}$$

where ρ and μ are respectively the fluid density and viscosity. The velocity $\mathbf{u}'$ is assumed to satisfy the no-slip boundary conditions on the surfaces of the spheres:

$$\mathbf{u}' = \mathbf{\Omega}^{A\prime} \times \mathbf{r}' \quad \text{on} \quad |\mathbf{r}'| = a, \tag{2.3a}$$

$$\mathbf{u}' = (\mathbf{U}'_B - \mathbf{U}'_A) + \mathbf{\Omega}^{B\prime} \times (\mathbf{r}' - \mathbf{l}') \quad \text{on} \quad |\mathbf{r}' - \mathbf{l}'| = b, \tag{2.3b}$$

where $\mathbf{l}'$ is the position vector of the centre of sphere B.

In terms of dimensionless quantities defined by

$$\left.\begin{aligned} &\mathbf{r} = \frac{\mathbf{r}'}{a}, \quad \mathbf{u} = \frac{\mathbf{u}'}{U'_A}, \quad p = \frac{ap'}{\mu U'_A}, \quad \mathbf{l} = \frac{\mathbf{l}'}{a}, \\ &\mathbf{U} = -\frac{\mathbf{U}'_A}{U'_A}, \quad \Delta\mathbf{U} = \frac{\mathbf{U}'_B - \mathbf{U}'_A}{U'_A}, \quad \mathbf{\Omega}^A = \frac{a\mathbf{\Omega}'_A}{U'_A}, \quad \mathbf{\Omega}^B = \frac{a\mathbf{\Omega}'_B}{U'_A}, \\ &R = \frac{\rho a U'_A}{\mu}, \quad \lambda = \frac{b}{a}, \quad \text{where} \quad U'_A \equiv |\mathbf{U}'_A|, \end{aligned}\right\} \tag{2.4}$$

(2.1)–(2.3) may be written as

$$\nabla^2\mathbf{u}-\nabla p = R\mathbf{u}.\nabla\mathbf{u}, \tag{2.5a}$$

$$\nabla.\mathbf{u} = 0, \tag{2.5b}$$

$$\mathbf{u}\rightarrow\mathbf{U} \quad \text{as} \quad r\equiv|\mathbf{r}|\rightarrow\infty, \tag{2.6}$$

$$\mathbf{u} = \mathbf{\Omega}^A\times\mathbf{r} \quad \text{on} \quad r=1, \tag{2.7a}$$

$$\mathbf{u} = \Delta\mathbf{U}+\mathbf{\Omega}^B\times\mathbf{q} \quad \text{on} \quad q(\equiv|\mathbf{q}|)=\lambda, \tag{2.7b}$$

where $\mathbf{q} = \mathbf{r}-\mathbf{l}$.

Here we assume that $R \ll 1$ and the non-dimensional distance l ($\equiv|\mathbf{l}|$) between the sphere centres is much smaller than a/R so that sphere B is located in the inner region of expansion of sphere A (case II). On the other hand, in case I studied by Vassure & Cox (1977), the distance l is assumed to be much greater than a/R.

3. The inner and outer expansions

The inner expansions of $\mathbf{u}$ and p are of the form (Brenner & Cox 1963)

$$\mathbf{u} = \mathbf{u}_0(\mathbf{r})+R\,\mathbf{u}_1(\mathbf{r})+o(R), \tag{3.1a}$$

$$p = p_0(\mathbf{r})+R\,p_1(\mathbf{r})+o(R). \tag{3.1b}$$

Substituting these into (2.5)–(2.7), and equating powers of R yields

$$\nabla^2\mathbf{u}_0-\nabla p_0 = 0, \tag{3.2a}$$

$$\nabla.\mathbf{u}_0 = 0, \tag{3.2b}$$

$$\mathbf{u}_0 = \mathbf{\Omega}^A\times\mathbf{r} \quad \text{on} \quad r=1, \tag{3.3a}$$

$$\mathbf{u}_0 = \Delta\mathbf{U}+\mathbf{\Omega}^B\times\mathbf{q} \quad \text{on} \quad q=\lambda, \tag{3.3b}$$

and

$$\nabla^2\mathbf{u}_1-\nabla p_1 = \mathbf{u}_0.\nabla\mathbf{u}_0, \tag{3.4a}$$

$$\nabla.\mathbf{u}_1 = 0, \tag{3.4b}$$

$$\mathbf{u}_1 = 0 \quad \text{on} \quad r=1 \quad \text{and} \quad q=\lambda. \tag{3.5}$$

The outer expansions are of the form

$$\mathbf{u} = \mathbf{U}+R\tilde{\mathbf{u}}_1(\tilde{\mathbf{r}})+o(R), \tag{3.6a}$$

$$p = R^2\tilde{p}_1(\tilde{\mathbf{r}})+o(R^2), \tag{3.6b}$$

where $\tilde{\mathbf{r}} = R\mathbf{r}$. Substituting these into (2.5) and (2.6), one finds

$$\tilde{\nabla}.\tilde{\mathbf{u}}_1-\tilde{\nabla}\tilde{p}_1 = \mathbf{U}.\tilde{\nabla}\tilde{\mathbf{u}}_1, \tag{3.7a}$$

$$\tilde{\nabla}.\tilde{\mathbf{u}}_1 = 0, \tag{3.7b}$$

$$\tilde{\mathbf{u}}_1\rightarrow 0 \quad \text{as} \quad \tilde{r}\rightarrow\infty. \tag{3.8}$$

The force $\mathbf{f}'^\sigma$ and torque $\mathbf{t}'^\sigma$ on sphere σ (σ is A or B) are expanded as

$$\mathbf{f}^\sigma \equiv \frac{\mathbf{f}'^\sigma}{6\pi a\mu U'_A} = \mathbf{f}_0^\sigma+R\mathbf{f}_1^\sigma+o(R), \tag{3.9a}$$

$$\mathbf{t}^\sigma \equiv \frac{\mathbf{t}'^\sigma}{6\pi a^2\mu U'_A} = \mathbf{t}_0^\sigma+R\mathbf{t}_1^\sigma+o(R), \tag{3.9b}$$

where $\mathbf{f}_0^\sigma$ and $\mathbf{f}_1^\sigma$ ($\mathbf{t}_0^\sigma$ and $\mathbf{t}_1^\sigma$) are the non-dimensional forces (torques) due to $(\mathbf{u}_0, p_0)$ and $(\mathbf{u}_1, p_1)$ respectively.

4. Zeroth-order inner approximation ($\mathbf{u}_0$, p_0) and the matching condition for ($\mathbf{u}_1$, p_1)

Equation (3.6a) yields the matching condition

$$\mathbf{u}_0 \rightarrow \mathbf{U} \quad \text{as} \quad r \rightarrow \infty, \tag{4.1}$$

and ($\mathbf{u}_0, p_0$) can be obtained by solving (3.2) with (3.3), (4.1). Because of the linearity of the equations (3.2) and the boundary conditions (3.3) and (4.1), the zeroth-order force $\mathbf{f}_0^\sigma$ (σ is A or B) due to ($\mathbf{u}_0, p_0$) depends linearly on $\mathbf{U}$, $\mathbf{\Omega}^A$ and $\mathbf{\Omega}^B$, where $\Delta\mathbf{U}$ is assumed to be negligible. From this and the symmetry consideration, it is shown that $\mathbf{f}_0^\sigma$ is of the form

$$\mathbf{f}_0^\sigma = \alpha_1^\sigma \mathbf{U} + \alpha_2^\sigma (\mathbf{U}\,.\,\mathbf{l})\mathbf{l} + \beta^\sigma \mathbf{\Omega}^A \times \mathbf{l} + \gamma^\sigma \mathbf{\Omega}^B \times \mathbf{l}, \tag{4.2}$$

where α_1^σ, α_2^σ, β^σ and γ^σ are scalar functions of l and λ only.

It is known that $\mathbf{u}_0$ and p_0 are expanded for large r as

$$\mathbf{u}_0 = \mathbf{U} - \mathbf{S}(\mathbf{r})\,.\,\mathbf{f}_0 + O(r^{-2}), \tag{4.3a}$$

$$p_0 = -\frac{3}{2r^2}\hat{\mathbf{r}}\,.\,\mathbf{f}_0 + O(r^{-3}), \tag{4.3b}$$

where $\mathbf{f}_0 = \mathbf{f}_0^A + \mathbf{f}_0^B$, $\hat{\mathbf{r}} = \mathbf{r}/r$, and $\mathbf{S}$ is a tensor defined by

$$S_{ij}(\mathbf{r}) = \frac{3}{4}\frac{1}{r}(\delta_{ij} + \hat{r}_i \hat{r}_j). \tag{4.4}$$

The first-order outer approximation ($\tilde{\mathbf{u}}_1, \tilde{p}_1$) satisfying (3.7), (3.8) and properly matching with (4.3) is known to yield the matching condition

$$\mathbf{u}_1 \sim \frac{3}{16}\left[\mathbf{U}\,.\,\nabla\left\{r\left(-3\mathbf{f}_0 + \frac{(\mathbf{r}\,.\,\mathbf{f}_0)\mathbf{r}}{r^2}\right)\right\} + \{3\mathbf{f}_0 - (\mathbf{U}\,.\,\mathbf{f}_0)\mathbf{U}\}\right] \quad \text{as} \quad r \rightarrow \infty, \tag{4.5a}$$

$$p_1 = o(r^{-1}) \quad \text{as} \quad r \rightarrow \infty \tag{4.5b}$$

(see Brenner & Cox 1963).

5. First-order force and torque

To calculate the first-order force $\mathbf{f}_1^A$ and torque $\mathbf{t}_1^A$ on sphere A due to ($\mathbf{u}_1, p_1$), it is convenient to introduce a Stokes field ($\mathbf{u}^*, p^*$) defined by

$$\nabla^2\mathbf{u}^* - \nabla p^* = 0, \quad \nabla\,.\,\mathbf{u}^* = 0, \tag{5.1a, b}$$

$$\mathbf{u}^* \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty, \tag{5.2a}$$

$$\left.\begin{aligned} \mathbf{u}^* &= \mathbf{V}^* + \mathbf{W}^* \times \mathbf{r} \quad \text{on} \quad r = 1, \\ \mathbf{u}^* &= 0 \qquad\qquad\qquad\; \text{on} \quad q = \lambda. \end{aligned}\right\} \tag{5.2b}$$

Equations (3.4a) and (5.1a) may be written as

$$(\tau_1)_{ij,j} = [(u_0)_i\,(u_0)_j]_{,j}, \tag{5.3}$$

$$(\tau^*)_{ij,j} = 0, \tag{5.4}$$

where $(\tau_1)_{ij}$ and $(\tau^*)_{ij}$ are the stress tensors due to ($\mathbf{u}_1, p_1$) and ($\mathbf{u}^*, p^*$) respectively. By noting $(u_1)_{i,j}\,(\tau^*)_{ij} = (u^*)_{i,j}\,(\tau_1)_{ij}$, it is shown that

$$[(u^*)_i\,(\tau_1)_{ij} - (u_1)_i\,(\tau^*)_{ij} - (u_0)_i\,(u_0)_j\,(u^*)_i]_{,j} + (u_0)_i\,(u_0)_j\,(e^*)_{ij} = 0, \tag{5.5}$$

where $(e^*)_{ij} = \frac{1}{2}\{(u^*)_{i,j} + (u^*)_{j,i}\}$ is the dimensionless rate-of-strain tensor for the flow $\mathbf{u}^*$. Now we integrate this over the volume V_L bounded externally by the spherical surface S_L of radius L, and internally by the surfaces $S_A(r = 1)$ and $S_B(q = \lambda)$. Then using the boundary conditions (3.5), (5.2*b*), and letting $L \to \infty$, we obtain

$$6\pi\{(V^*)_i (f_1^A)_i + (W^*)_i (t_1^A)_i\} = \int_{S_A} (u^*)_i (\tau_1)_{ij} \, dS_j = I_1 - I_2 - I_3 + I, \tag{5.6}$$

where

$$I_1 = \lim_{L\to\infty} \int_{S_L} (u^*)_i (\tau_1)_{ij} \, dS_j, \tag{5.7}$$

$$I_2 = \lim_{L\to\infty} \int_{S_L} (u_1)_i (\tau^*)_{ij} \, dS_j, \tag{5.8}$$

$$I_3 = \lim_{L\to\infty} \int_{S_L} (u_0)_i (u_0)_j (u^*)_i \, dS_j, \tag{5.9}$$

$$I = \lim_{L\to\infty} \int_{V_L} (u_0)_i (u_0)_j (e^*)_{ij} \, dV, \tag{5.10}$$

and $d\mathbf{S}$ has the direction of the outer normal to the volume bounded by the surface S_A or S_L. It is shown in a manner following Brenner & Cox (1963) that

$$I_1 = 0, \tag{5.11a}$$

$$I_2 = \tfrac{9}{8}\pi\{3(f_0)_i - U_i (f_0)_j U_j\} (f^*)_i, \tag{5.11b}$$

$$I_3 = \lim_{L\to\infty} \int_{S_L} U_i U_j (u^*)_i \, dS_j, \tag{5.11c}$$

where

$$6\pi(f^*)_i \equiv \lim_{L\to\infty} \int_{S_L} (\tau^*)_{ij} \, dS_j = 6\pi\{(f_A^*)_i + (f_B^*)_i\}, \tag{5.12a}$$

$$6\pi(f_A^*)_i = \int_{S_A} (\tau^*)_{ij} \, dS_j, \tag{5.12b}$$

$$6\pi(f_B^*)_i = \int_{S_B} (\tau^*)_{ij} \, dS_j. \tag{5.12c}$$

6. Evaluation of I when $a/l' \ll 1$ and $\mathbf{W}^* = 0$

When $\epsilon \equiv 1/l \ll 1$, applying the method of reflection, we obtain

$$\mathbf{u}_0 = \mathbf{U} - \mathsf{D}(\mathbf{r}) \, . \, \mathbf{h}^A + \mathsf{T}(\mathbf{r}) \, . \, \mathbf{\Omega}^A - \mathsf{D}\left(\frac{\mathbf{q}}{\lambda}\right) . \, \mathbf{h}^B + \mathsf{T}\left(\frac{q}{\lambda}\right) . \, \mathbf{\Omega}^B + O(\epsilon^2), \tag{6.1}$$

where $\Delta\mathbf{U}$ in (3.3*b*) is assumed to be negligible, the matrices D and T are defined by

$$D_{ij}(r) = S_{ij}(\mathbf{r}) + \tfrac{1}{6}\nabla^2 S_{ij}(\mathbf{r}), \tag{6.2}$$

$$T_{ij}(\mathbf{r}) = \epsilon_{ijk} \frac{\hat{r}_k}{r^2}, \tag{6.3}$$

and

$$\mathbf{h}^A = \mathbf{U} - S\left(\frac{-\mathbf{l}}{\lambda}\right) . \, \mathbf{U}, \tag{6.4a}$$

$$\mathbf{h}^B = \mathbf{U} - S(\mathbf{l}) \, . \, \mathbf{U}. \tag{6.4b}$$

Similarly, we obtain

$$\mathbf{u}^* = \mathbf{D}(\mathbf{r}) \cdot \mathbf{V}^* + \mathbf{T}(\mathbf{r}) \cdot \mathbf{W}^* - \mathbf{D}\left(\frac{\mathbf{q}}{\lambda}\right) \cdot \mathbf{S}(\mathbf{l}) \cdot \mathbf{V}^* + O(\epsilon^2). \tag{6.5}$$

To obtain the force $(f_1^A)_i$ we put $(V^*)_p = \delta_{pi}$ and $\mathbf{W}^* = 0$ in (5.6). Then, substituting (6.1) and (6.5) into (5.6) and taking into account the relations $S_{ij}(-\mathbf{r}) = S_{ij}(\mathbf{r})$, $T_{ij}(-\mathbf{r}) = -T_{ij}(\mathbf{r})$, we obtain

$$I = I_3 + I_A + I_B - 4\pi\lambda U_j U_k J^1_{ijk} + 2\pi\epsilon [U_j U_k \{2\lambda J^2_{ijk} + \lambda^2 J^3_{ijk}\} + 2U_j \Omega^B_k \lambda^2 J^4_{ijk}] + O(\epsilon^2), \tag{6.6}$$

where

$$I_A = \lim_{L\to\infty} \int_{L>r>1} \left[U_j - D_{jp}(\mathbf{r}) h^A_p - D_{jp}\left(\frac{-\mathbf{l}}{\lambda}\right) h^B_p\right] \times T_{kq}(\mathbf{r}) \Omega^A_q \{D_{ji,k}(\mathbf{r}) + D_{ki,j}(\mathbf{r})\} d^3\mathbf{r}, \tag{6.7}$$

$$I_B = -\lim_{L\to\infty} \int_{L>q>\lambda} \left[U_j - D_{jp}\left(\frac{\mathbf{q}}{\lambda}\right) h^B_p - D_{jp}(\mathbf{l}) h^A_p\right] \times \left[T_{kq}\left(\frac{\mathbf{q}}{\lambda}\right)\Omega^B_q\right]\left[D_{jr,k}\left(\frac{\mathbf{q}}{\lambda}\right) + D_{kr,j}\left(\frac{\mathbf{q}}{\lambda}\right)\right] S_{ri}(\mathbf{l})\, d^3\mathbf{r}, \tag{6.8}$$

$$(2\pi) J^m_{ijk} = \lim_{\substack{\delta\to 0\\ L\to\infty}} \int_{\substack{L>\bar{r}>\delta\\ \bar{q}>\lambda\delta}} L^m_{ijk}\, d^3\bar{\mathbf{r}}, \tag{6.9}$$

in which

$$\lambda L^1_{ijk} = S_{pk}\left(\frac{\bar{\mathbf{q}}}{\lambda}\right) E^i_{pj}(\bar{\mathbf{r}}), \tag{6.10}$$

$$\lambda L^2_{ijk} = S_{pj}(\bar{\mathbf{r}}) S_{qk}\left(\frac{\bar{\mathbf{q}}}{\lambda}\right) E^i_{pq}(\bar{\mathbf{r}}), \tag{6.11}$$

$$\lambda^2 L^3_{ijk} = S_{pj}\left(\frac{\bar{\mathbf{q}}}{\lambda}\right) S_{pk}\left(\frac{\bar{\mathbf{q}}}{\lambda}\right) E^i_{pq}(\bar{\mathbf{r}}), \tag{6.12}$$

$$\lambda^2 L^4_{ijk} = T_{pk}\left(\frac{\bar{\mathbf{q}}}{\lambda}\right) E^i_{jp}(\bar{\mathbf{r}}), \tag{6.13}$$

$$E^i_{jk}(\mathbf{r}) = \tfrac{1}{2}\{S_{ji,k}(\mathbf{r}) + S_{ki,j}(\mathbf{r})\} = \frac{3}{4r^2}\hat{r}_i(\delta_{jk} - 3\hat{r}_j\hat{r}_k) \tag{6.14}$$

and $\bar{\mathbf{r}} = \mathbf{r}/l$, $\bar{\mathbf{g}} = \mathbf{g}/l$. It is shown in the appendix that

$$\left.\begin{aligned} I_A &= \pi(\mathbf{h}^A \times \mathbf{\Omega}^A)_i + O(\epsilon^2),\\ I_B &= -\pi\lambda^2[\mathbf{S}(\mathbf{l}) \cdot (\mathbf{U} \times \mathbf{\Omega}^B)]_i + O(\epsilon^2). \end{aligned}\right\} \tag{6.15}$$

Now let us take the coordinate system in which

$$\mathbf{l} = l\mathbf{e}_1, \quad \mathbf{U} = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2, \tag{6.16}$$

where $\cos\theta = \mathbf{U} \cdot \mathbf{l}$, and $\mathbf{e}_i$ is the unit vector in the direction of the ith Cartesian component. Then J^m_{ijk} with $m \leqslant 3$ and j or $k = 3$, and J^4_{i3k} do not appear in (6.6). Some of the other Js are easily shown to be zero. To calculate the remaining Js, it is convenient to use a formula for a bipolar integral:

$$\int f(k,p,|\mathbf{k}-\mathbf{p}|)\, d^3\mathbf{p} = \int f(k,p,r)\,\delta^3(\mathbf{k}-\mathbf{p}-\mathbf{r})\, d^3\mathbf{p}\, d^3\mathbf{r} = \int_\triangle \frac{2\pi pr}{k} f(k,p,r)\, dp\, dr, \tag{6.17}$$

where the $\triangle$ indicates that the integration is restricted to the part of the (p, r)-plane in which k, p, r can be the sides of a triangle. The formula (6.17) is known to be useful also in the study of turbulence (Leslie 1973). The calculation was carried out with the algebraic manipulation language REDUCE-2 (Hern 1973) in the Computer Centre of the University of Tokyo. The following rational numbers in (6.18) are obtained by such a computation manipulated algebraically, and not by a deduction from decimal numbers. The results are as follows:

$$\left.\begin{aligned} &J^1_{111} = -\tfrac{9}{16}, \quad J^1_{122} = \tfrac{9}{32}, \quad J^1_{221}+J^1_{212} = -\tfrac{9}{16}, \\ &J^2_{111} = -\tfrac{27}{32}, \quad J^2_{122} = \tfrac{27}{128}, \quad J^2_{221}+J^2_{212} = -\tfrac{27}{128}, \\ &J^3_{111} = -\tfrac{27}{16}, \quad J^3_{122} = \tfrac{27}{32}, \quad J^3_{221}+J^3_{212} = -\tfrac{27}{32}, \\ &J^4_{123} = \tfrac{3}{4}, \quad J^4_{321} = -\tfrac{3}{4}. \end{aligned}\right\} \tag{6.18}$$

The other Js are zero or do not appear in (6.6). The above value for each J was checked to be in good agreement with (within a difference less than 10^{-4} from) the result of numerical integration obtained by using a library program AQ2DD in the Computer Centre of Nagoya University.

7. The force on sphere A

For $V^*_p = \delta_{pi}$ and $\mathbf{W}^* = 0$, we have

$$\mathbf{f}^*_A = -\mathbf{e}_i + O(\epsilon^2), \tag{7.1}$$

$$\mathbf{f}^*_B = \lambda \mathbf{S}(\mathbf{l}) \,.\, \mathbf{e}_i + O(\epsilon^3). \tag{7.2}$$

From (5.6), (5.11), (6.6), (7.1) and (7.2), the first-order force $\mathbf{f}^A_1$ is given by

$$\begin{aligned} 6\pi (f^A_1)_i = {} & \tfrac{3}{8}\pi\{(3\mathbf{f}_0 - (\mathbf{U} \,.\, \mathbf{f}_0)\mathbf{U}) \,.\, (\mathbf{e}_i - \lambda \mathbf{S}(\mathbf{l}) \,.\, \mathbf{e}_i)\} + \pi\{\mathbf{h}^A \times \mathbf{\Omega}^A \\ & - \lambda^2 \mathbf{S}(\mathbf{l}) \,.\, (\mathbf{U} \times \mathbf{\Omega}^B)\}_i - 4\pi\lambda U_p U_q J^1_{ipq} + 2\pi\epsilon\{U_p U_q (2\lambda J^2_{ipq} \\ & + \lambda^2 J^3_{ipq}) + 2\lambda^2 U_p \Omega^B_k J^4_{ipk}\} + O(\epsilon^2), \end{aligned}$$

where p and q take only the values 1 and 2.

The following observations can be made from this result.

(i) If $\mathbf{U}$ is perpendicular to $\mathbf{l}(\theta = \frac{1}{2}\pi)$ or $\mathbf{U} = (0, 1, 0)$, the force $(f'^A)_1$ on sphere A and parallel to $\mathbf{l}$ is

$$\begin{aligned} (f'^A)_1 = 6\pi\mu a U'_A\{R[& -\tfrac{3}{16}\lambda + \tfrac{1}{6}\Omega^A_3 + \epsilon(\tfrac{9}{64}\lambda + \tfrac{9}{32}\lambda^2 \\ & -\tfrac{1}{8}\lambda\,\Omega^A_3 + \tfrac{1}{4}\lambda^2\,\Omega^B_3) + O(\epsilon^2)] + o(R)\}. \end{aligned} \tag{7.4}$$

When $R = 0$, the force f' given by (7.4) is clearly zero. If the rotational velocities of the spheres are very small, $(f'^A)_1$ is negative, so that two spheres experience repulsive forces.

It is to be noted that the rotation of sphere B affects $(f'^A)_1$ in (7.4), as well as that of sphere A. In the limit $\lambda \to 0$, (7.4) reduces to

$$(f'^A)_1 = \pi a \mu U \Omega^A_3 (R + o(R)) \tag{7.5}$$

in accordance with the result for a single rotating sphere obtained by Rubinow & Keller (1961).

It is also observed that the leading term in (7.4) has no dependence on l. In case I, i.e. when $l \gg a/R$, according to Vassure & Cox (1977),

$$(f'^A)_1 = 6\pi\mu a U'_A\left\{-\tfrac{3}{4}\epsilon^2 \frac{1}{R}\left[2 - \left(\frac{R}{\epsilon} + 2\right)\exp\left(-\frac{R}{2\epsilon}\right)\right]\right\}, \tag{7.6}$$

for $a = b$ $(\lambda = 1)$ and $\mathbf{\Omega}^A = \mathbf{\Omega}^B = 0$.

In the limit $R/\epsilon \to 0$ this yields

$$(f'^A)_1 = 6\pi\mu a U'_A(-\tfrac{3}{16}R), \tag{7.7}$$

which is in agreement with (7.4) for $\mathbf{\Omega}^A = 0$ and $a/l \to 0$.

(ii) If $\mathbf{U}$ is parallel to $\mathbf{l}(\theta = 0$ or $\pi)$, then $(f_0)_1 = ((1+\lambda)-3\epsilon\lambda)\,U_1 + O(\epsilon^2)$, and the first-order force $(f_1'^A)_1$ on sphere A and parallel to $\mathbf{U}$ is

$$(f_1'^A)_1 = 6\pi\mu a U'_A R\{\tfrac{3}{8}(1+\lambda-3\epsilon\lambda)\,(1-\tfrac{3}{2}\epsilon\lambda)\,U_1 + \tfrac{3}{8}\lambda + \epsilon(-\tfrac{9}{16}\lambda - \tfrac{9}{16}\lambda^2) + O(\epsilon^2)\}, \tag{7.8}$$

This force (up to $O(\epsilon)$) is not affected by the rotations of spheres. If sphere A is in the leading position, i.e. $\theta = 0$ or $\mathbf{U} = (1, 0, 0)$, then

$$(f'^A)_1 = 6\pi\mu a U'_A\{(f_0^A)_1 + R\,[\tfrac{3}{8} + \tfrac{3}{4}\lambda - \epsilon(\tfrac{9}{4}\lambda + \tfrac{9}{8}\lambda^2) + O(\epsilon^2)] + o(R)\}, \tag{7.9}$$

while if it is in the trailing position, i.e. $\theta = \pi$, or $\mathbf{U} = (-1, 0, 0)$, then

$$(f'^A)_1 = 6\pi\mu a U'_A\{(f_0^A)_1 + R\,[-\tfrac{3}{8} + \epsilon(\tfrac{9}{8}\lambda) + O(\epsilon^2)] + o(R)\}, \tag{7.10}$$

where $(f_0^A)_1$ is the force obtained by the analysis based on the Stokes equations. Thus the drag on a sphere is larger when it is in the leading position than when it is in the trailing position. The difference Δ_f between the drag in these two cases is

$$\Delta_f(\lambda) = 6\pi\mu a U'_A\{R\,[\tfrac{3}{4}\lambda - \tfrac{9}{8}\epsilon(\lambda+\lambda^2) + O(\epsilon^2)] + o(R)\}, \tag{7.11}$$

From the above result it can be shown that if the two spheres are of the same radius a then the leading sphere experiences larger drag than the trailing one and the difference between the drags is $\Delta_f(1)$, i.e. they effectively experience an attractive force of magnitude $\Delta_f(1)$.

In case I ($lR/a \ll 1$), for $a = b$

$$\Delta_f = -6\pi\mu a U'_A\left\{\frac{3}{2}\frac{\epsilon^2}{R}\left[1-\exp\left(-\frac{R}{\epsilon}\right)\right] - \frac{3}{2}\epsilon\right\} + \dots. \tag{7.12}$$

$$\Delta_f \to 6\pi\mu a U'_A \tfrac{3}{4} R \quad \text{as} \quad R/\epsilon \to 0,$$

which is in accordance with (7.11) for $\epsilon \to 0$.

Appendix. Derivation of (6.15)

First consider I_A, which may be written as

$$I_A = I_{A1} - I_{A2} + O(\epsilon^2), \tag{A 1}$$

$$I_{A1} = h_j^A\,\Omega_q^A \int_{r>1} d^3\mathbf{r}\,\epsilon_{kqm}\frac{1}{r^2}\hat{r}_m X_{ijk}(\mathbf{r}), \tag{A 2}$$

$$I_{A2} = h_p^A\,\Omega_q^A \int_{r>1} d^3\mathbf{r}\,\epsilon_{kqm} D_{jp}(\mathbf{r})\frac{1}{r^2}\hat{r}_m X_{ijk}(\mathbf{r}), \tag{A 3}$$

where

$$D_{jp}(\mathbf{r}) = \frac{1}{4}\left[\frac{3}{r}(\delta_{jp} + \hat{r}_j\hat{r}_p) + \frac{1}{r^3}(\delta_{jp} - 3\hat{r}_j\hat{r}_p)\right], \tag{A 4}$$

$$\begin{aligned} X_{ijk}(\mathbf{r}) &\equiv D_{ji,k}(\mathbf{r}) + D_{ki,j}(\mathbf{r}) \\ &= \frac{3}{2}\left[\frac{1}{r^2}(\hat{r}_i\,\delta_{jk} - 3\hat{r}_i\,\hat{r}_j\,\hat{r}_k) - \frac{1}{r^4}(\hat{r}_i\,\delta_{jk} + \hat{r}_j\,\delta_{ki} + \hat{r}_k\,\delta_{ij} - 5\hat{r}_i\,\hat{r}_j\,\hat{r}_k)\right]. \end{aligned} \tag{A 5}$$

Substituting (A 5) into (A 2), (A 3), and noting that $\epsilon_{kqm}\hat{r}_m\hat{r}_k = 0$, we have

$$I_{A1} = h_j^A \Omega_q^A \epsilon_{kqm} \int_1^\infty dr\, 4\pi r^2 \left\langle \frac{\hat{r}_m}{r^2}\frac{3}{2}\left(\frac{\hat{r}_i\delta_{jk}}{r^2} - \frac{\hat{r}_i\delta_{jk}+\hat{r}_j\delta_{ki}}{r^4}\right)\right\rangle_r, \tag{A 6}$$

$$I_{A2} = h_p^A \Omega_q^A \epsilon_{kqm} \int_1^\infty dr\, 4\pi r^2 \left\langle D_{jp}(\mathbf{r})\frac{\hat{r}_m}{r^2}\frac{3}{2}\left(\frac{\hat{r}_i\delta_{jk}}{\hat{r}^2} - \frac{\hat{r}_i\delta_{jk}+\hat{r}_j\delta_{ki}}{\hat{r}^4}\right)\right\rangle_r, \tag{A 7}$$

where $\langle\ \ \rangle_r$ denotes the average over the spherical surface $r = R$. Using $\langle\hat{r}_i\hat{r}_j\rangle_r = \frac{1}{3}\delta_{ij}$, we have

$$\begin{aligned} I_{A1} &= h_j^A \Omega_q^A \epsilon_{kqm}\, 2\pi\{\delta_{im}\delta_{jk} - \tfrac{1}{3}(\delta_{im}\delta_{jk}+\delta_{jm}\delta_{ki})\} = 2\pi h_j^A \Omega_q^A \epsilon_{jqi} \\ &= 2\pi(\mathbf{h}^A \times \boldsymbol{\Omega}^A)_i, \end{aligned} \tag{A 8}$$

$$\begin{aligned} I_{A2} &= h_p^A \Omega_q^A \epsilon_{kqm}\, 2\pi\tfrac{1}{4}\{(\tfrac{3}{2}+\tfrac{1}{4}-\tfrac{3}{4}-\tfrac{1}{6})\delta_{im}\delta_{kp} - (\tfrac{3}{4}+\tfrac{3}{4}+\tfrac{1}{6}-\tfrac{3}{6})\delta_{mp}\delta_{ki}\} = \pi h_p^A \Omega_q^A \epsilon_{pqi} \\ &= \pi(\mathbf{h}^A \times \boldsymbol{\Omega}^A)_i. \end{aligned} \tag{A 9}$$

Thus
$$I_A = \pi(\mathbf{h}^A \times \boldsymbol{\Omega}^A)_i + O(\epsilon^2). \tag{A 10}$$

As for I_B, putting $\mathbf{p} = \mathbf{q}/\lambda$, we have

$$I_B = -\lambda^3 \int_{p>1} d^3\mathbf{p}\, [h_d^B - D_{jp}(\mathbf{p})\, h_p^B][T_{kq}(\mathbf{p})\,\Omega_q^B]\frac{1}{\lambda}\left[\frac{\partial}{\partial p_k}D_{jr}(\mathbf{p}) + \frac{\partial}{\partial p_j}D_{kr}(\mathbf{p})\right] S_{ri}(\mathbf{l}) + O(\epsilon^2). \tag{A 11}$$

From (A 2), (A 3), (A 8), (A 9),

$$\begin{aligned} I_B &= -\pi\lambda^2 S_{ri}(\mathbf{l})\{(2\mathbf{h}^B - \mathbf{h}^B)\times\boldsymbol{\Omega}^B\}_r + O(\epsilon^2) \\ &= -\pi\lambda^2\{\mathbf{S}(\mathbf{l})\,.\,(\mathbf{h}^B \times \boldsymbol{\Omega}^B)\}_i + O(\epsilon^2), \end{aligned} \tag{A 12}$$

where the relation $S_{ri}(\mathbf{l}) = S_{ri}(\mathbf{l})$ has been used.

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